A curious variation of the figure-eight orbit

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Abstract

A set of stable figure-eight orbits for three stars, which together librate back and forth through almost pi/2 radians, and also precess through a full circle.

1. Introduction

The figure eight orbit was discovered numerically by Cristopher Moore [1], and later analytically by Alain Chenciner & Richard Montgomery [2], and further examined by Carles Simó [3] and many others. This interesting variation, in which the shapes of the orbits librate back and forth through an angle of π/2 radians, is by Bob Jenkins [4].

This paper examines and confirms the behaviour found by [4], and adds some further details.

The following XML file contains initial parameters for the standard figure-eight orbit, but with one value amended, which replicates the amendment made by [4]. This gives one of the stars a small z component in its initial position by setting pZ=0.1, all other values remaining the same.


```
<system>
  <body>
    <mass>1</mass>
    <pX>-0.970004362</pX>
    <pY>0.24308753</pY>
    <pZ>0</pZ>
    <vX>-0.466203685</vX>
    <vY>-0.43236573</vY>
    <vZ>0</vZ>
  </body>
  <body>
    <mass>1</mass>
    <pX>0.970004362</pX>
    <pY>-0.24308753</pY>
    <pZ>0</pZ>
    <pZ>0.1</pZ>
    <vX>-0.466203685</vX>
    <vY>-0.43236573</vY>
    <vZ>0</vZ>
  </body>
  <body>
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    <pX>0</pX>
    <pY>0</pY>
    <pZ>0</pZ>
    <vX>0.93240737</vX>
    <vY>0.86473146</vY>
    <vZ>0</vZ>
  </body>
</system>
```

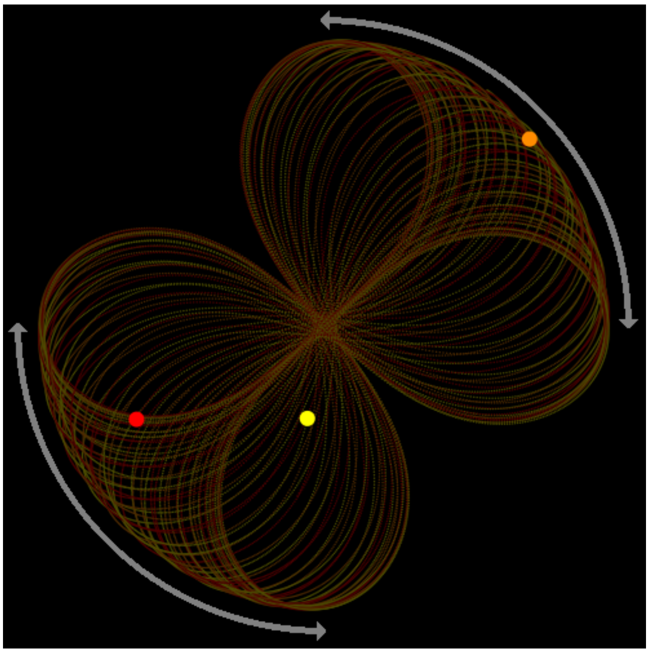

Without this amended value, the parameters would produce the standard figure-eight orbit, in which the all three stars share a single orbit which remains on a constant fixed plane. However, the amendment, which gives one of the stars a small z component in its initial position, introduces a three-dimensional aspect to the orbits.

2. Libration of the shapes of the orbits through almost π/2 radians

The remarkable effect of this amendment is examined first viewed from the z axis, so that the motions of the three stars in the x and y directions are visible.

While the three stars are moving along their orbits, the shapes of their orbits turn together around the z axis, first slowly, then faster, then slowly again, ceasing when the shapes of the orbits have completed an angle of about π/2 radians. The shapes of the orbits then turn together in the opposite sense, first slowly, then faster, then slowly, ceasing when when they have got back to their original orientation of 0 radians. This cycle repeats, the shapes of the orbits librating together back and forth through an angle of almost π/2 radians, see Figure 1.

Figure 1: Traced motions in the x and y directions during the first libration period

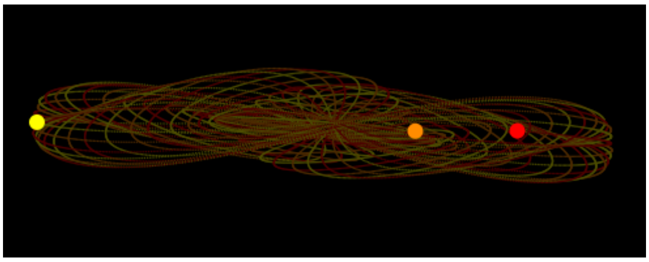


A remarkable observation is made by [4], which is that for various values of pZ within a reasonable range, the libration angle is always the same. The libration angle does not vary with pZ.

The angle of libration was found to be approximatly 0.47π radians (surprisingly a little less than 0.5π radians).

The orbits of the three stars are not confined to the xy plane, as shown in Figure 2.

Figure 2: Edge-on view of traced motions in the x and z directions during the first libration period



3. The libration period

For pZ=0.1, the period of the libration is equal to about 11.5 orbital periods.
The axis of libration is initially the z axis (however, this axis changes, as will be described later in this paper).

Experiments were made to estimate how the libration period varies with different values of pZ, and the results are listed in Figure 3, where the unit used for time is the orbital period .

Figure 3: Libration period for various values of pZ of an initially non-central star

Initial pZ (of an initially non-central star)	Libration period (approximate)
0.2	5.5
0.1	11.5
0.05	23
0.025	46

These values were estimated visually from a simulator display and are therefore very approximate. However they do suggest that (within this range of pZ values) the libration period may be inversely proportional to pZ.

In the starting parameters, one of the stars is initially positioned at the origin. The other two stars are initially mirrors of each other, in that the initial position and velocity of one may be obtained by simply multiplying the initial position and velocity components of the other by minus one. In the above example the pZ value is given to one of these two initially non-central stars. Giving a pZ value to the other initially non-central star instead produces exactly the same libration angle and libration period, because they are mirrors of each other in the starting parameters.

If however the pZ value is instead given to the initially central star, the effect is in one important respect different. A similar libration occurs, with the same libration angle, however the libration period remarkably is halved, as can be seen by comparing Figures 3 and 4.

Figure 4: Libration period for various values of pZ of the initially central star

Initial pZ (of the initially central star)	Libration period (approximate)
0.1	5.5
0.05	11.5

4. Precession of the axis of libration

Now there is a further twist. In addition to the libration of the shapes of the orbits, there is also a longer-term continuous precession of the axis of libration. For pZ=0.1, the period of this precession is about 1350 orbital periods. Which is equal to about 117 libration periods.

The axis (around which this long-term precession of the axis of libration occurs) is a fixed line in the xy plane, which in Figure 1 is approximately diagonal, going from top-left, through the origin, to bottom-right, The angle between this axis of precession and the y axis is approximately 0.235π radians (a little less than 0.25π radians).

So the shapes of the orbits librate back and forth through about 0.47π radians around a libration axis which initially (at exactly t=0) is equal to the z axis, but which thereafter slowly precesses so that it is no longer equal to the z axis.

In Figures 5 and 6 and 7 the motions of the stars are traced for a complete precession period, viewed from the z axis, viewed from the y axis, and viewed from the x axis.

Figure 5: Traced motions for approx one complete precession period viewed from the **z** axis

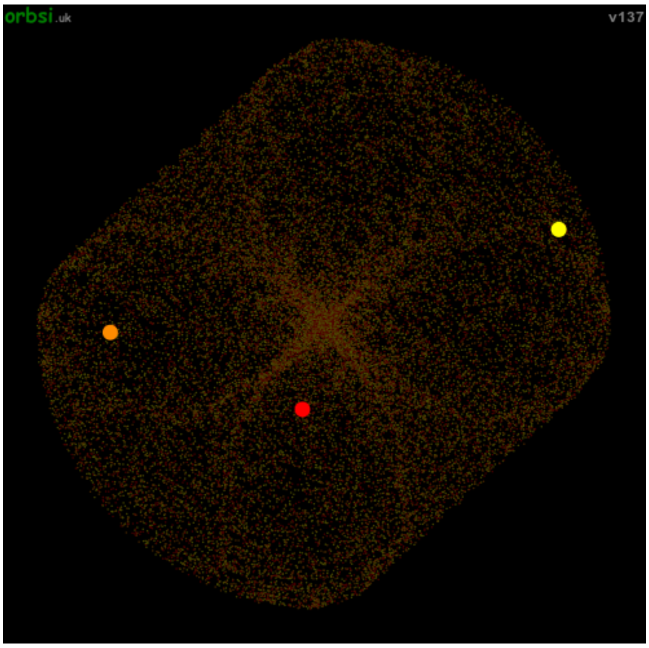


Figure 6: Traced motions for approx one complete precession period viewed from the **y** axis

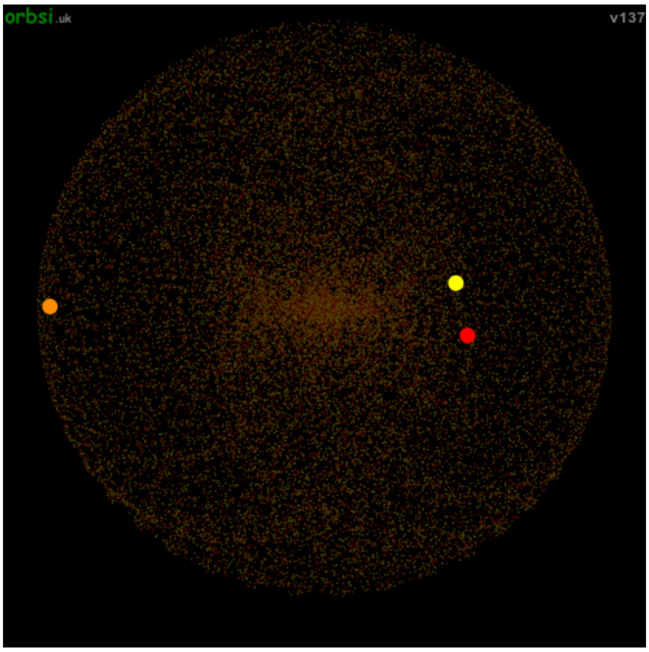
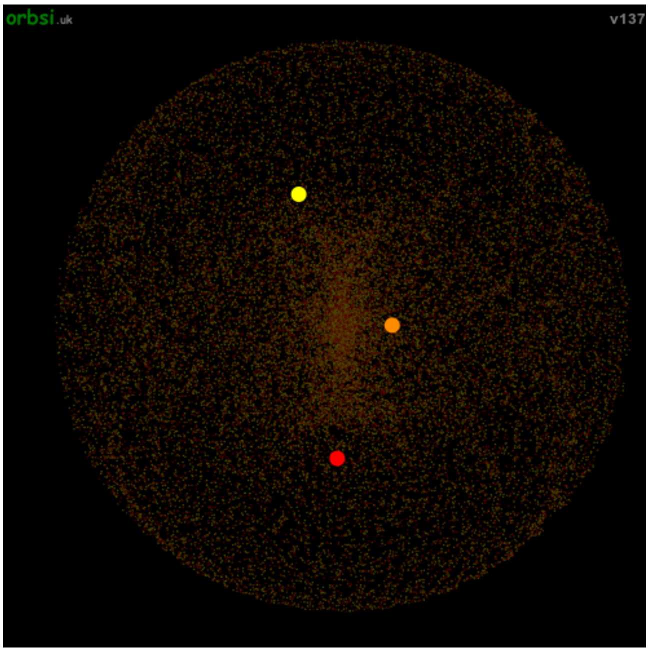


Figure 7: Traced motions for approx one complete precession period viewed from the **x** axis



Therefore in a complete precession period the motions of the stars completely fill a three-dimensional shape which may be described as follows:

Starting with a solid sphere, slice its top off at a latitude slightly greater than 0.235π radians, slice its base off at a latitude slightly less than - 0.235π radians, round the corners where it was sliced, then make two rounded conical depressions (one in the top surface and one in the bottom surface).

After one complete precession period, comprising many libration periods, the system returns to its starting configuration, and commences the same cycle again.

5. The precession period

For pZ=0.1, the period of this precession is about 1350 orbital periods. Experiments were made to investigate how the precession period varies with different values of pZ.

This variation of the figure-eight seems to always have one more surprise in store. The fairly reasonable expectation, that the precession period might be related in a simple way to the libration period, was soon abandoned. Because when the value of pZ applied to an initially non-central star was increased from pZ=0.1 to pZ=0.2, it was found that the sense of the precession is reversed: it surprisingly goes in the opposite direction.

Therefore the equation which shows how the precession period varies with pZ is not found here, except to say that it will be rather complicated.

6. A special case with no precession

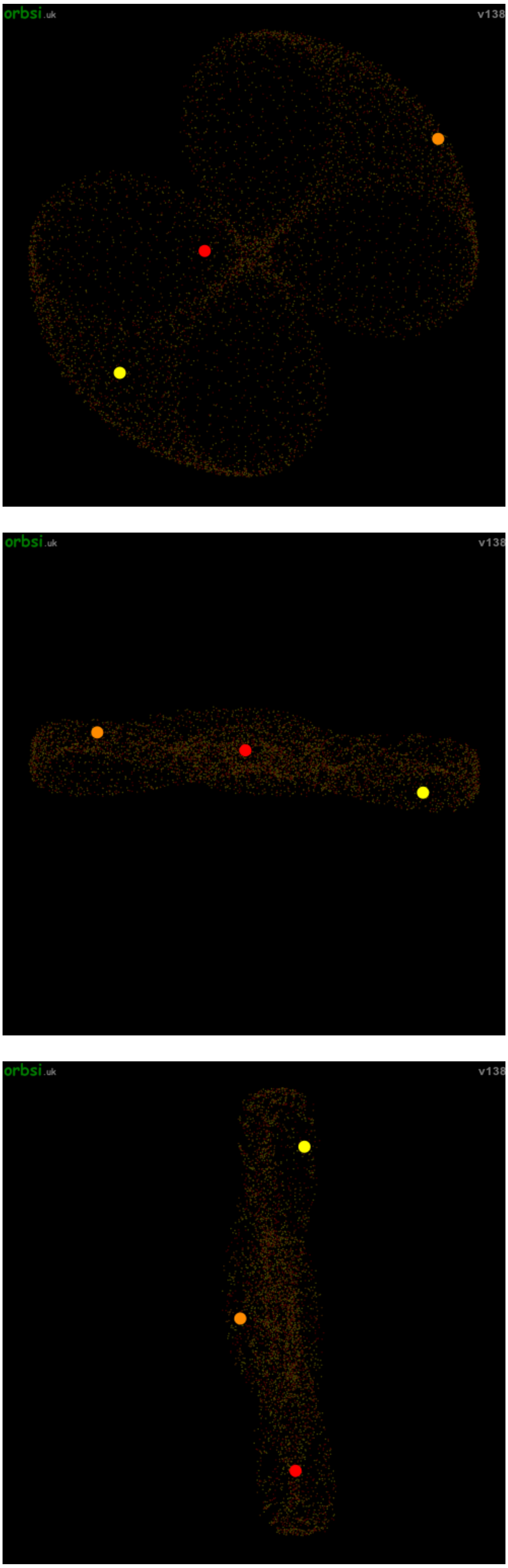
However this experiment did give another rather interesting result as follows. The observation, that the precessions for pZ=0.2 and for pZ=0.1 go in opposite directions, implies that there may be some other value of pZ, intermediate between 0.1 and 0.2, which results in zero precession.

Experiments to find this intermediate value were successful and it was found that the value which gives zero precession is approximately pZ=0.186. When this value of pZ is given to one of the initially non-central stars, the libration back and forth through about 0.47π radians still occurs, but the precession of the axis of libration is close to zero, and so the axis of libration remains approximately equal to the z axis.

For this special value of approximately pZ=0.186, the three-dimensional shape, which is filled over time by the motions of the stars, differs greatly that described in section 4 above.

It is a peculiar lozenge shape which is best described by the 3 views in Figure 8, which are from the z axis (top), from the y axis (middle), and from the x axis.

Figure 8: Traced motions over time for the non-precessing value pZ=0.186, from z axis, y axis, x axis



7. Simulations

Online orbit simulations are available [5] [6] of the general case with pZ=0.1, and of the special case with pZ=0.186.

8. Conclusions

A variation of the figure-eight orbit, which was described previously by [4], was examined.

The results obtained by [4] were confirmed to be accurate, as follows:

A. If one of the stars in the standard figure-eight orbit is given a small out-of-plane position component pZ, then the shapes of the 3 orbits jointly librate back and forth through an angle of almost $\pi/2$ radians.

B. The libration angle does not vary with pZ. The libration angle is the same for all reasonably small non-zero values of pZ.

C. The pZ value may be applied to any one of the three stars.

D. Additionally there is a continuous precession, through 2π radians, of the axis of libration.

The following further details were discovered:

E. For reasonably small values of pZ, the libration period is approximately inversely proportional to pZ.

F. Applying a given pZ to the initially central star (compared with applying it to either of the initially non-central stars) produces the same libration angle but produces a libration period of half the duration.

G. For pZ=0.1, the libration period is equal to about 11.5 orbital periods and the period of precession of the axis of libration is equal to about 1350 orbital periods.

H. The libration angle is estimated to be approximatly 0.47π radians (surprisingly, a little less than 0.5π radians).

I. The relationship between the precession period and pZ is rather complicated (for example the precession produced by pZ=0.2 goes in the opposite direction compared with the precession produced by pZ=0.1).

J. There is a special case, produced by a value of approximately pZ=0.186, in which the precession is zero and therefore the axis of libration remains fixed in space.

9. Addendum (09 March 2015)

This addendum adds some further information.

(i) By taking the triple-star system defined by the XML file listed above, and rotating its initial state around the z axis by -0.235π radians, a more accurate illustration of the three-dimensional shape of the orbits is produced here, and some clearer simulations.

(ii) The value of pZ which, when applied to an initially non-central star, results in zero precession, is now more accurately estimated as pZ=0.189.

(iii) The improved illustration of the three-dimensional shape filled by the orbits is shown in Figure 9. It traces the orbits for the special case where pZ=0.189 (zero precession of the axis of libration). The main view shows the orbits traces viewed from the z axis. Also shown are (at the top) the view from the y axis, and (on the right) the view from the x axis.

(iv) The three-dimensional shape filled by the orbits may be visualised by mentally combining the 3 views in Figure 9. It is a strange 3D shape which is two-lobed (in different ways) whether viewed from the z axis, the y axis, or the x axis.

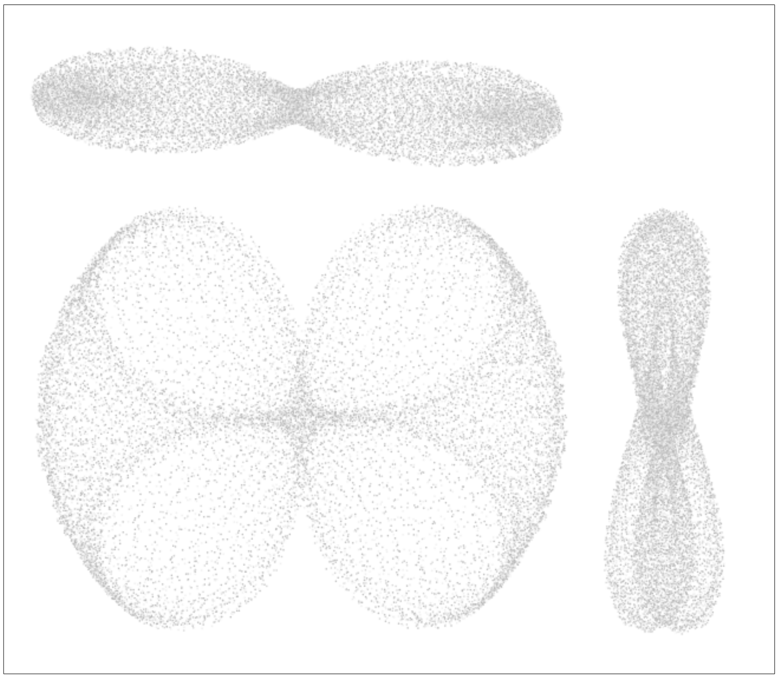
(v) In the special case where there is zero precession (pZ=0.189), this three-dimensional shape, which encloses the orbits, remains fixed in space.

(vi) For other values of pZ, the three dimensional shape is approximately the same (except that it may be thicker or thinner in the z direction), however it does not remain fixed in space. The three-dimensional shape (together with the orbits it encloses) slowly rotates (precesses) around its y axis. By the time it has rotated by a complete circle, it has filled the three-dimension shape of much larger volume which was described in in the main part of this paper.

(vii) The improved online simulations [7] [8] show the system rotated so that the libration is symmetrical about the xz plane, and the axis of precession is equal to the y axis, which makes the simulations clearer. The first simulation shows an example of the general case where there is both libration and precession. The second simulation shows the special case where there is libration but almost no precession.

(viii) Values of pZ exceeding about 0.22 were found to be unstable. And of course if pZ is set to zero, it produces the standard figure-eight orbit.

Figure 9: The three-dimensional shape filled by a librating figure-eight orbit with pZ=0.189



10. **Further addendum** (12 April 2015)

Since writing the above, I learned of two interesting papers [9] [10] which describe rotating variations of the figure-eight orbit. See in particular Figure 12 in [10] which shows a variation similar to the variations illustrated here.

11. References

[1] Moore C
Braids in classical dynamics
Phys. Rev. Lett. 70, 3675 (1993)
tuvalu.santafe.edu/~moore/braids-prl.pdf

[2] Chenciner A, Montgomery R
A remarkable periodic solution of the three-body problem...
Annals of Mathematics 152 (2000) 881.
www.emis.de/journals/Annals/152_3/chencine.pdf

[3] Simó C
New Families of Solutions in N-Body Problems
www.math.uni-bielefeld.de/~rehmann/ECM/cdrom/3ecm/pdfs/pant3/simo.pdf

[4] Jenkins B
Figure Eight Orbits
burtleburtle.net/bob/physics/
The "figure-eight orbits" page uses java applets.
The relevant simulation is the sixth simulation from the top, labelled "+.1z".

[5] Online simulation of an example of the general case (librating and precessing), with pZ=0.1
www.orbsi.uk/space/simulator/simulator.php?s=00051

[6] Online simulation of the special case pZ=0.186 (librating but with almost zero precession)
www.orbsi.uk/space/simulator/simulator.php?s=00052

[7] Improved online simulation of an example of the general case (librating and precessing), with pZ=0.1
www.orbsi.uk/space/simulator/simulator.php?s=00053

[8] Improved online simulation of the special case pZ=0.189 (librating but with almost zero precession)
www.orbsi.uk/space/simulator/simulator.php?s=00054

[9] Chenciner A, Fejoz J, Montgomery R
Rotating Eights: I. The three Γi families
Nonlinearity, Mar 2005; 18(3):1407
www.ceremade.dauphine.fr/~fejoz/Articles/rotatingEights.pdf

[10] Nauenberg N
Continuity and stability of families of figure eight orbits with finite angular momentum
Celestial Mechanics and Dynamical Astronomy , Jan 2007, Vol 97, Issue 1, pp 1-15
physics.ucsc.edu/~michael/marc9.pdf

Version history:
v.1. 08 March 2015
v.2 08 March 2015 (reworded, added more illustrations, added description of the special case)
v.3 09 March 2015 (corrected the libration angle, and added the triple Figure 8)
v.4 17 March 2015 (updated links to improved n-body simulator)
v.5 24 March 2015 (improved description of the 3D shape filled by the orbits over a complete precession period)
v.6 12 April 2015 (merged addendum into main paper, and added two references).
v.7 29 Nov 2017 (corrected one simulatorlink).